Fixing and distinguishing numbers of trees

Calum Buchanan

UNIVERSITY OF DENVER

AMS Spring Sectional at Boston College
Special Session on Recent Trends in Graph Theory

March 28, 2026

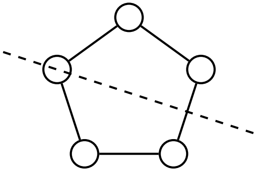
Joint work with Peter Dankelmann, Isabel Harris, Paul Horn, K. E. Perry, and Emily Rivett-Carnac

Symmetry breaking

This talk is about symmetries in simple graphs.

An *automorphism* of $G = (V, E)$ is a bijection

$$\varphi : V \to V \qquad \text{such that} \qquad uv \in E \iff \varphi(u)\varphi(v) \in E.$$



Symmetry breaking

This talk is about symmetries in simple graphs.

An *automorphism* of $G = (V, E)$ is a bijection

$$\varphi : V \to V \qquad \text{such that} \qquad uv \in E \iff \varphi(u)\varphi(v) \in E.$$

Question

What is the minimum amount of extra information needed to pick out a graph from the others in its automorphism class?

Symmetry breaking

This talk is about symmetries in simple graphs.

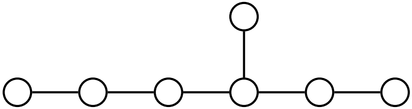
An *automorphism* of $G = (V, E)$ is a bijection

$$\varphi : V \to V \qquad \text{such that} \qquad uv \in E \iff \varphi(u)\varphi(v) \in E.$$

Question

What is the minimum amount of extra information needed to pick out a graph from the others in its automorphism class?

Immediately identifiable (asymmetric):



Symmetry breaking

This talk is about symmetries in simple graphs.

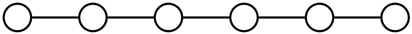
An *automorphism* of $G = (V, E)$ is a bijection

$$\varphi : V \to V \qquad \text{such that} \qquad uv \in E \iff \varphi(u)\varphi(v) \in E.$$

Question

What is the minimum amount of extra information needed to pick out a graph from the others in its automorphism class?

Not immediately identifiable:



Symmetry breaking

This talk is about symmetries in simple graphs.

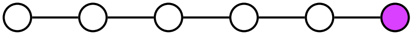
An *automorphism* of $G = (V, E)$ is a bijection

$$\varphi : V \to V \qquad \text{such that} \qquad uv \in E \iff \varphi(u)\varphi(v) \in E.$$

Question

What is the minimum amount of extra information needed to pick out a graph from the others in its automorphism class?

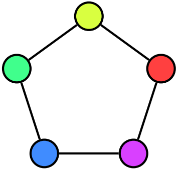
Not immediately identifiable:



Distinguishing numbers

A *distinguishing coloring* of a graph G is a coloring of its vertices which is fixed only by the identity automorphism of G.

Example:



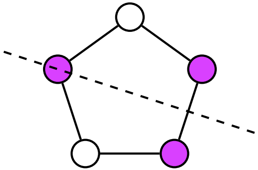
History:

- Babai (1977) under the name “asymmetric coloring"
- Albertson and Collins (1996)

Distinguishing numbers

A *distinguishing coloring* of a graph G is a coloring of its vertices which is fixed only by the identity automorphism of G.

Nonexample:



History:

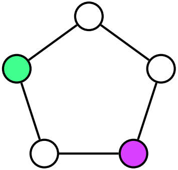
- Babai (1977) under the name “asymmetric coloring"
- Albertson and Collins (1996)

Distinguishing numbers

A *distinguishing coloring* of a graph G is a coloring of its vertices which is fixed only by the identity automorphism of G.

The *distinguishing number* $D(G)$ is the minimum number of colors needed.

Example: $D(C_5) = 3$



History:

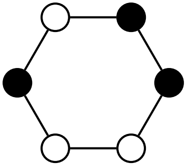
- Babai (1977) under the name “asymmetric coloring"
- Albertson and Collins (1996)

Distinguishing numbers

A *distinguishing coloring* of a graph G is a coloring of its vertices which is fixed only by the identity automorphism of G.

The *distinguishing number* $D(G)$ is the minimum number of colors needed.

Example: $D(C_6) = 2$



History:

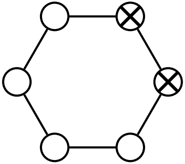
- Babai (1977) under the name "asymmetric coloring"
- Albertson and Collins (1996)

Fixing numbers

A *fixing set* of a graph G is a subset of its vertices which is fixed pointwise only by the identity automorphism of G.

The *fixing number* $F(G)$ is the minimum cardinality of a fixing set.

Example: $F(C_6) = 2$



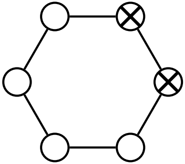
History:

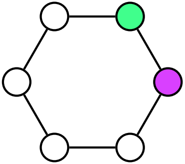
- Harary (1996, 2001)
- Boutin (2006) under the name “determining number"
- Fijavž and Mohar (2004) under the name “rigidity index"

Observations

Proposition (Albertson and Boutin (2007))

For any graph G, we have $D(G) \leqslant F(G) + 1$.

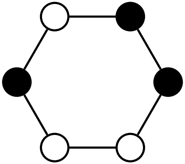


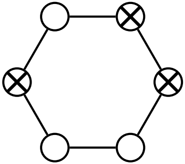


Observations

Proposition

For any graph G of order n, we have $F(G) \leqslant \frac{D(G)-1}{D(G)} n$.

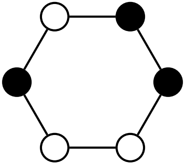


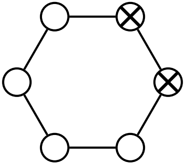


Observations

Proposition

For any graph G of order n, we have $F(G) \leqslant \frac{D(G)-1}{D(G)} n$.
If at least two color classes have more than one vertex, the inequality is strict.

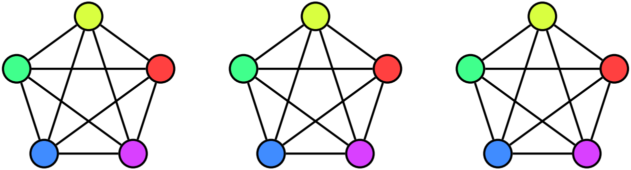




Observations

Proposition

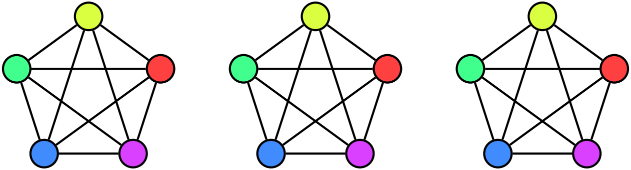
For any graph G of order n, we have $F(G) \leqslant \frac{D(G)-1}{D(G)} n$. This bound is sharp.



Observations

Proposition

For any graph G of order n, we have $F(G) \leqslant \frac{D(G)-1}{D(G)} n$. This bound is sharp.



Question

What is the maximum fixing number of a D-distinguishable tree?

Trees

Theorem (Erwin and Harary (2006))

For every tree T, there is a minimum cardinality fixing set consisting only of leaves of T.

Trees

Theorem (Erwin and Harary (2006))

For every tree T, there is a minimum cardinality fixing set consisting only of leaves of T.

$O(n \log n)$-time algorithms to compute $D(T)$ by Cheng (2006); Arvind and Devanur (2004)

Theorem (Tymoczko (2004))

For every tree T with maximum degree Δ, $D(T) \leqslant \Delta$.

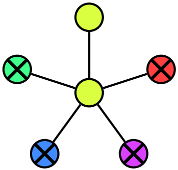
Characterization of trees with distinguishing number two and radius at most three by Alikhani and Soltani (2016)

Maximum fixing number of a D-distinguishable tree

Theorem (BDHHPR (2026))

For any n-vertex tree T with distinguishing number $D \geqslant 3$, we have $F(T) \leqslant \frac{D-1}{D+1}n$. This bound is sharp.

Example ($D = 5$)

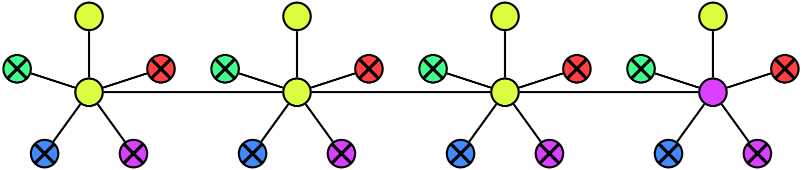


Maximum fixing number of a D-distinguishable tree

Theorem (BDHHPR (2026))

For any n-vertex tree T with distinguishing number $D \geqslant 3$, we have $F(T) \leqslant \frac{D-1}{D+1} n$. This bound is sharp.

Example ($D = 5$)

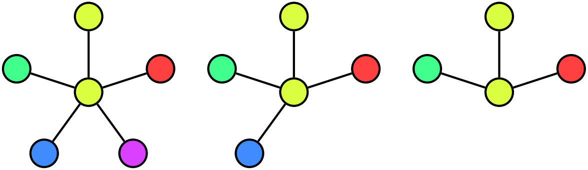


Maximum fixing number of a D-distinguishable tree

Theorem (BDHHPR (2026))

For any n-vertex tree T with distinguishing number $D \geqslant 3$, we have $F(T) \leqslant \frac{D-1}{D+1}n$. This bound is sharp.

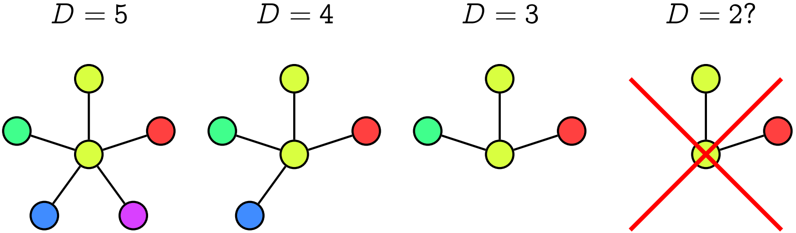
$D = 5$ $D = 4$ $D = 3$



Maximum fixing number of a D-distinguishable tree

Theorem (BDHHPR (2026))

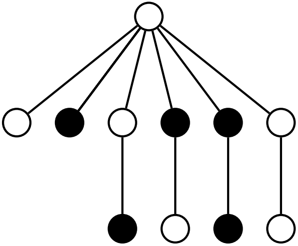
For any n-vertex tree T with distinguishing number $D \geqslant 3$, we have $F(T) \leqslant \frac{D-1}{D+1} n$. This bound is sharp.



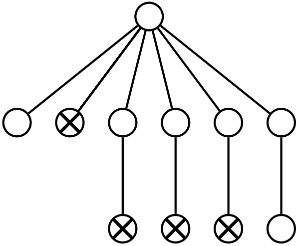
Maximum fixing number of a 2-distinguishable tree

Theorem (BDHHPR (2026))

For any 2-distinguishable tree T of order $n \geqslant 3$, we have $F(T) \leqslant \frac{4}{11}n$. This bound is sharp.



A distinguishing 2-coloring

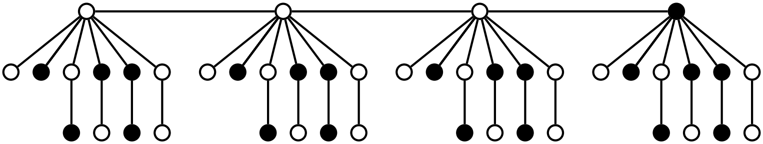


A minimum fixing set

Maximum fixing number of a 2-distinguishable tree

Theorem (BDHHPR (2026))

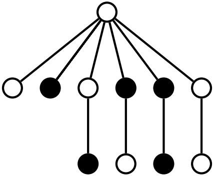
For any 2-distinguishable tree T of order $n \geqslant 3$, we have $F(T) \leqslant \frac{4}{11}n$. This bound is sharp.



Idea of proof of 4/11 theorem

Lemma

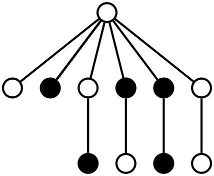
There is a unique 2-distinguishable spider (tree with one high degree vertex) of maximum fixing density.



Idea of proof of 4/11 theorem

Lemma

There is a unique 2-distinguishable spider (tree with one high degree vertex) of maximum fixing density.



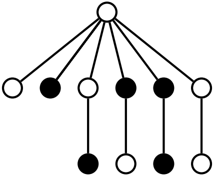
For a D-distinguishable spider T, let $n_k(T)$ denote the number of legs (branches) of length k. Then $F(T) = \sum \max\{0, n_k - 1\}$.

- $n_k(T) \leqslant D^k$ for all k.

Idea of proof of 4/11 theorem

Lemma

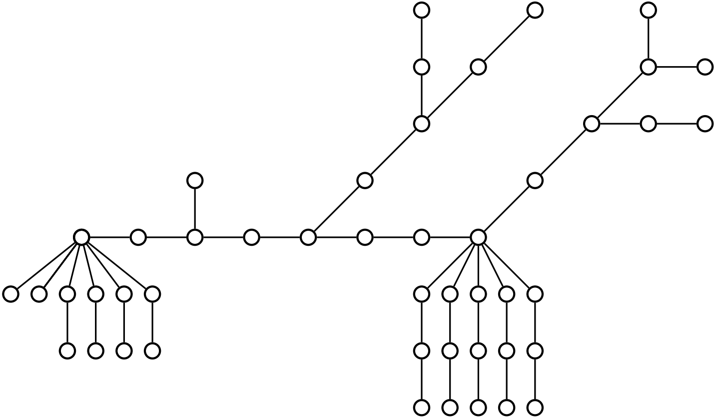
There is a unique 2-distinguishable spider (tree with one high degree vertex) of maximum fixing density.



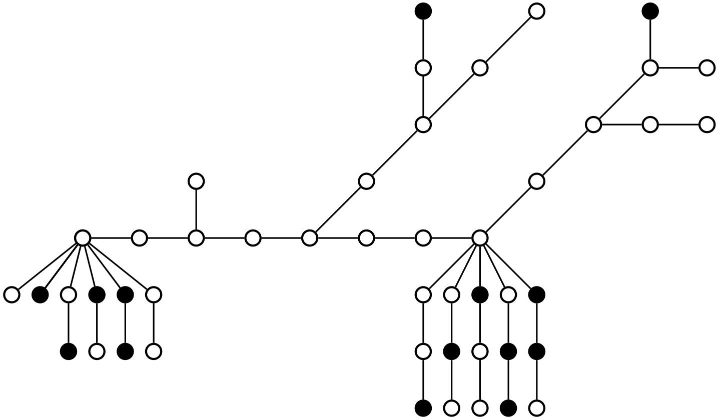
For a D-distinguishable spider T, let $n_k(T)$ denote the number of legs (branches) of length k. Then $F(T) = \sum \max\{0, n_k - 1\}$.

- $n_k(T) \leqslant D^k$ for all k.
- If $n_k(T) \neq 0$ for some k, then there is a spider T' whose fixing density is at least as large as T, so that $n_k(T') \in \{0, D^k\}$ with $n_i(T') = n_i(T)$ for $i \neq k$.

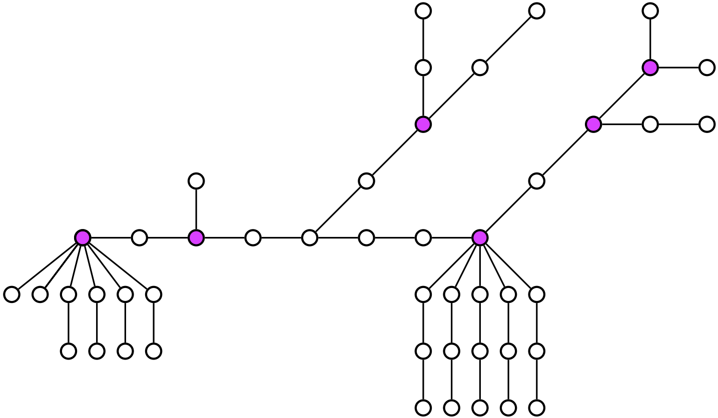
Idea of proof of 4/11 theorem



Idea of proof of 4/11 theorem

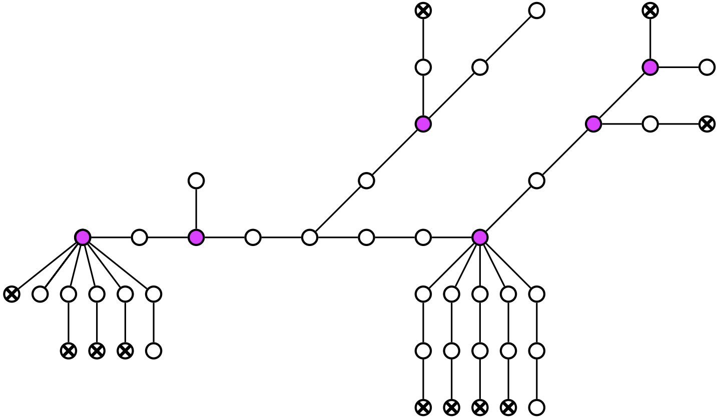


Idea of proof of 4/11 theorem



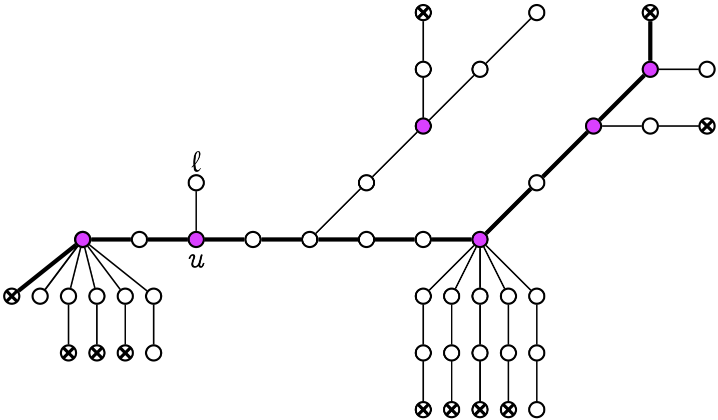
All leaves contained in (2-distinguishable) pendent spiders.

Idea of proof of 4/11 theorem



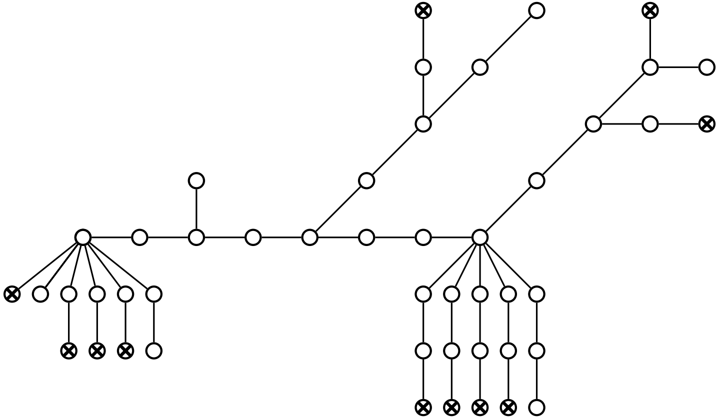
Pendent spiders S of order $|S| \geqslant 3$ can be fixed internally with $4|S|/11$ vertices, all leaves.

Idea of proof of 4/11 theorem



Vertex ℓ is the only leaf not contained in a pendent spider S with $|S| \geqslant 3$, but u is fixed by every nontrivial automorphism which fixes all leaves in such S.

Idea of proof of 4/11 theorem



If all leaves are fixed by an automorphism, it must be the identity. Thus, the above is a fixing set of density $\leqslant 4/11$.

Future Work

Question

What is the maximum fixing number of a connected D-distinguishable graph of order n?

Future Work

Question

What is the maximum fixing number of a connected D-distinguishable graph of order n?

A family of constructions with $F(G) = \frac{D-1}{D}(n - O(\log n))$:

- H a D-distinguishable graph of order k
- For each nonempty $U \subseteq V(H)$, add D new vertices and join these to U
- $|G| = k + D(2^k - 1)$ and $F(G) = (D-1)(2^k - 1)$

Future Work

Question

What is the maximum fixing number of a connected D-distinguishable graph of order n?

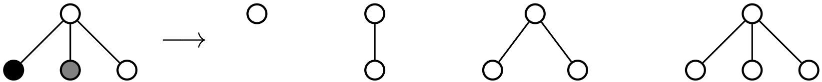
A family of constructions with $F(G) = \frac{D-1}{D}(n - O(\log n))$:

- H a D-distinguishable graph of order k
- For each nonempty $U \subseteq V(H)$, add D new vertices and join these to U
- $|G| = k + D(2^k - 1)$ and $F(G) = (D-1)(2^k - 1)$

Are there connected graphs with $F(G) = \frac{D-1}{D}(n - o(\log n))$?

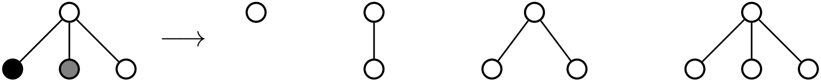
What do D-distinguishable trees look like?

Radius $\leqslant 1$: branched subgraphs of $K_{1,D}$

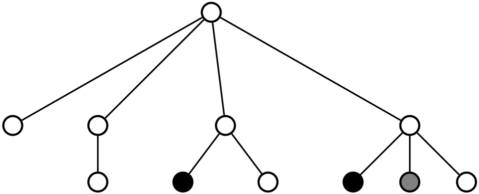


What do D-distinguishable trees look like?

Radius $\leqslant 1$: branched subgraphs of $K_{1,D}$

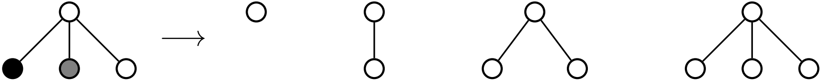


Radius $\leqslant 2$: branched subgraphs of $K_{1,D}$ appended to a new root

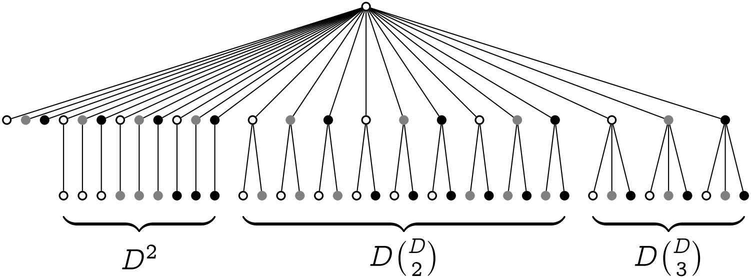


What do D-distinguishable trees look like?

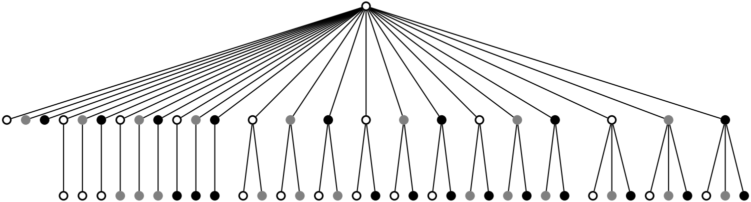
Radius $\leqslant 1$: branched subgraphs of $K_{1,D}$



Radius $\leqslant 2$: *copies of* branched subgraphs of $K_{1,D}$ appended to a new root



Universal trees



Theorem (BDHHPR (2026))

Let D and r be positive integers with $D \geqslant 2$. There exists a rooted tree T_r^D which contains all D-distinguishable trees of radius at most r as branched subgraphs. Further, every branched subgraph of T_r^D having radius $r \geqslant 2$ is D-distinguishable.

Universal trees

Theorem (BDHHPR (2026))

Let D and r be positive integers with $D \geqslant 2$. There exists a rooted tree T_r^D which contains all D-distinguishable trees of radius at most r as branched subgraphs. Further, every branched subgraph of T_r^D having radius $r \geqslant 2$ is D-distinguishable.

Question

Let T_{D_n} be a universal tree which contains all D-distinguishable trees of order n, $D \geqslant 1$. What is the minimum number of edges in T_{D_n}?

Thank you!

References I

Michael O. Albertson and Debra L. Boutin. Using determining sets to distinguish Kneser graphs. *Electron. J. Combin.*, 14:R20, 2007.

Michael O. Albertson and Karen L. Collins. Symmetry breaking in graphs. *Electron. J. Combin.*, 3(1):R18, 1996. ISSN 1077-8926.

Saeid Alikhani and Samaneh Soltani. Trees with distinguishing number two. *AKCE International Journal of Graphs and Combinatorics*, 16, 11 2016. doi: 10.1016/j.akcej.2018.02.001.

Vikraman Arvind and Nikhil Devanur. Symmetry breaking in trees and planar graphs by vertex coloring. *Proceedings of the Nordic Combinatorial Conference*, 10 2004.

L. Babai. Asymmetric trees with two prescribed degrees. *Acta Math. Acad. Sci. Hungar.*, 29(1-2):193–200, 1977. doi: 10.1007/BF01896481.

Debra L. Boutin. Identifying graph automorphisms using determining sets. *Electron. J. Combin.*, 13(1):R78, 2006.

Calum Buchanan, Peter Dankelmann, Isabel Harris, Paul Horn, K. E. Perry, and Emily Rivett-Carnac. On fixing and distinguishing numbers of trees, 2026. URL `https://arxiv.org/abs/2603.23820`.

Christine T. Cheng. On computing the distinguishing numbers of trees and forests. *Electron. J. Combin.*, 13(1):R11, 2006.

References II

David Erwin and Frank Harary. Destroying automorphisms by fixing nodes. *Discrete Math.*, 306(24):3244–3252, 2006. doi: 10.1016/j.disc.2006.06.004.

Gašper Fijavž and Bojan Mohar. Rigidity and separation indices of paley graphs. *Discrete Math.*, 289(1-3):157–161, 2004.

F Harary. Survey of methods of automorphism destruction in graphs, invited address. In *Eighth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications, Kalamazoo, Michigan*, page 16, 1996.

Frank Harary. Methods of destroying the symmetries of a graph. *Bull. Malays. Math. Sci. Soc. (2)*, 24(2): 183–191, 2001. ISSN 0126-6705.

Julianna Tymoczko. Distinguishing numbers for graphs and groups. *Electron. J. Combin.*, 11(1):R63, 2004. ISSN 1077-8926.